

GORENSTEINNESS, HOMOLOGICAL INVARIANTS AND GORENSTEIN DERIVED CATEGORIES

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ABSTRACT. Relations between Gorenstein derived categories, Gorenstein defect categories and Gorenstein stable categories are established. Using these, the Gorensteinness of an algebra A and invariants with respect to recollements of the bounded Gorenstein derived category $D_{gp}^b(A\text{-mod})$ of A are investigated. Specifically, the Gorensteinness of A is characterized in three ways: the existence of Auslander-Reiten triangles in $D_{gp}^b(A\text{-mod})$; recollements of $D_{gp}^b(A\text{-mod})$; and also Gorenstein derived equivalences. It is shown that the finiteness of Cohen-Macaulay type and of finitistic dimension are invariant with respect to the recollements of $D_{gp}^b(A\text{-mod})$.

Key words: Gorenstein-projective modules; algebras of finite Cohen-Macaulay type; virtually Gorenstein algebras; Gorenstein derived categories; Gorenstein defect categories; Gorenstein stable categories.

1. Introduction

The Gorensteinness of an algebra is of interest in the representation theory of algebras, in Gorenstein homological algebra, and in the theory of singularity categories (see e.g. [Ha2], [CV], [Be2], [BJO], [K], [LZ]). How to characterize Gorenstein property is a basic problem.

An algebra has many invariants, for example, the finiteness of global dimension, the finiteness of finitistic dimension, and the finiteness of Cohen-Macaulay type (see e.g. [Wi], [Ha4], [Be4]). How to describe and compare these homological invariants is a major topic of interest.

One can approach these questions by derived (and related) categories, as well as comparisons of derived categories. Derived categories, introduced by Grothendieck and Verdier([Ver]), have been playing an increasingly important role in various areas of mathematics, including representation theory, algebraic geometry, and mathematical physics. For example, Happel ([Ha3]) has shown a finite dimensional algebra A has finite global dimension if and only if the bounded derived category $D^b(A\text{-mod})$ has Auslander-Reiten triangles. There are two ways to compare derived categories. One way is by derived equivalences. Happel ([Ha1]) has shown the finiteness of global dimension of an algebra is invariant under derived equivalences. Another way is by recollements, which have been introduced by Beilinson, Bernstein and Deligne ([BBD]). A recollement of a derived category by another two derived categories is a diagram of six functors between these

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categories, generalising Grothendieck's six functors. Suppose that A , B and C are three finite dimensional algebras over a field. If $D^b(A\text{-mod})$ admits a recollement with respect to the bounded derived categories $D^b(B\text{-mod})$ and $D^b(C\text{-mod})$ of B and C , Wiedemann ([Wi]) has shown that A has finite global dimension if and only if B and C have so, and Happel ([Ha4]) has shown that A has finite finitistic dimension if and only if B and C have so.

For Gorenstein homological algebra we refer to [AM, Buc, CFH, EJ, Hol]. Nan Gao and Pu Zhang defined the corresponding version of the derived category in Gorenstein homological algebra. They introduced in [GZ] the notions of Gorenstein derived category and Gorenstein derived equivalence which are needed in this context. Following [GZ], the bounded Gorenstein derived category $D_{gp}^b(A\text{-mod})$ of an algebra A is defined as the Verdier quotient of the bounded homotopy category $K^b(A\text{-mod})$ with respect to the triangulated subcategory $K_{gpac}^b(A\text{-mod})$ of $A\text{-}\mathcal{G}proj$ -acyclic complexes. Later, Gao ([G2]) described Auslander-Reiten triangles in the bounded Gorenstein derived categories for Gorenstein algebras of finite CM-type. In [G3] a sufficient and necessary criterion has been given for the existence of recollements of Gorenstein derived categories. Based on these work, two questions arise.

- (1) Can we characterize the Gorensteinness of an algebra in terms of the corresponding Gorenstein derived category?
- (2) Which invariants can be compared along recollements of Gorenstein derived categories?

In this article we will provide answers to these questions. Our answer to question (1) are the combination of Corollary 3.2 and 3.3, and Theorem 3.4 and 3.9(2). We state them as Theorem A.

Theorem A Let A be an artin algebra. Consider the following statements:

- (1) A is Gorenstein;
- (2) A is virtually Gorenstein and $D_{gp}^b(A\text{-mod})$ has Auslander-Reiten triangles;
- (3) A is virtually Gorenstein, and there exist Gorenstein algebras B and C and a recollement

$$D_{gp}^b(B\text{-mod}) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} D_{gp}^b(A\text{-mod}) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} D_{gp}^b(C\text{-mod});$$

- (3') A is virtually Gorenstein, and for arbitrary virtually Gorenstein algebras B and C , if there exists the following recollement

$$D_{gp}^b(B\text{-mod}) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} D_{gp}^b(A\text{-mod}) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} D_{gp}^b(C\text{-mod}),$$

then B and C are Gorenstein;

- (4) There is a triangle-equivalence $D_{gp}^b(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}proj)$;
- (5) $D_{gp}^b(A\text{-mod})$ has Auslander-Reiten triangles;
- (6) A is Gorenstein derived equivalent to an algebra B , which is Gorenstein.

We have the following relations between these statements:

- (i) $(1) \iff (2) \iff (3) \iff (3')$
- (ii) $(1) \iff (6)$

(iii) If $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$, then (1) \iff (4).

(iv) If A is of finite CM-type, then (1) \iff (5).

Our answer to question (2) are the combination of Theorem 3.9(1) and 3.12. We state them as Theorem B.

Theorem B Let A , B and C be artin algebras. Assume that the bounded Gorenstein derived category $D_{gp}^b(A\text{-mod})$ admits a recollement with respect to $D_{gp}^b(B\text{-mod})$ and $D_{gp}^b(C\text{-mod})$. The following hold true:

(1) If A , B and C are virtually Gorenstein, then A is of finite CM-type if and only if B and C are so;

(2) If A , B and C are of finite CM-type, then $\text{fd}A < \infty$ if and only if $\text{fd}B < \infty$ and $\text{fd}C < \infty$.

Let us end this introduction by mentioning that in private communication with Javad Asadollahi, he points out that he and his collaborators also have proofs for Corollary 3.2 and Theorem 3.9 in this paper. Their proofs are obtained independently, and also are different from the proofs given in this paper. The author would like to thank him for letting us know their proofs.

2. Preliminaries

In this section we fix notation and recall the main concepts to be used.

Let A be an artin algebra. Denote by $A\text{-Mod}$ (resp. $A\text{-mod}$) the category of left A -modules (resp. the category of finitely-generated left A -modules), and $A\text{-Proj}$ (resp. $A\text{-proj}$) the full subcategory of projective A -modules (resp. the full subcategory of finitely-generated projective A -modules). An A -module M is said to be Gorenstein-projective in $A\text{-Mod}$ (resp. $A\text{-mod}$), if there is an exact sequence $P^\bullet = \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow P^2 \rightarrow \cdots$ in $A\text{-Proj}$ (resp. $A\text{-proj}$) with $\text{Hom}_A(P^\bullet, Q)$ exact for any A -module Q in $A\text{-Proj}$ (resp. $A\text{-proj}$), such that $M \cong \ker d^0$ (see [EJ]). Denote by $A\text{-}\mathcal{GP}$ (resp. $A\text{-}\mathcal{G}proj$) the full subcategory of Gorenstein-projective modules in $A\text{-Mod}$ (resp. $A\text{-mod}$), and similarly denote by $A\text{-}\mathcal{GI}$ the full subcategory of Gorenstein-injective modules in $A\text{-Mod}$.

A proper Gorenstein-projective resolution of A -module M in $A\text{-mod}$ is an exact sequence $E^\bullet = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ such that all $G_i \in A\text{-}\mathcal{G}proj$, and that $\text{Hom}_A(G, E^\bullet)$ stays exact for each $G \in A\text{-}\mathcal{G}proj$. The second requirement guarantees the uniqueness of such a resolution in the homotopy category (the Comparison Theorem; see [EJ], p.169).

The Gorenstein-projective dimension $\mathcal{G}pdim M$ of M in $A\text{-mod}$ is defined to be the smallest integer $n \geq 0$ such that there is an exact sequence $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with all $G_i \in A\text{-}\mathcal{G}proj$, if it exists; and $\mathcal{G}pdim M = \infty$ if there is no such exact sequence of finite length. For an A -module ${}_A X$ we denote by $\text{proj.dim}_A X$ the projective dimension of X . Clearly $\mathcal{G}pdim M \leq \text{proj.dim}_A M$. Denote by $\text{fGd}(A) = \sup\{\mathcal{G}pdim_A X \mid \mathcal{G}pdim_A X < \infty\}$.

A complex C^\bullet of finitely-generated A -modules is $A\text{-}\mathcal{G}proj$ -acyclic, if $\text{Hom}_A(G, C^\bullet)$ is acyclic for any $G \in A\text{-}\mathcal{G}proj$. It is also called proper exact for example in [AM]. A chain map $f^\bullet : X^\bullet \rightarrow Y^\bullet$ is an $A\text{-}\mathcal{G}proj$ -quasi-isomorphism, if $\text{Hom}_A(G, f^\bullet)$ is a

quasi-isomorphism for any $G \in A\text{-}\mathcal{G}proj$, i.e., there are isomorphisms of abelian groups $H^n \text{Hom}_A(G, f^\bullet) : H^n \text{Hom}_A(G, X^\bullet) \cong H^n \text{Hom}_A(G, Y^\bullet)$, $\forall n \in \mathbb{Z}$, $\forall G \in A\text{-}\mathcal{G}proj$.

Denote by $K^-(A\text{-}\mathcal{G}proj)$ the upper bounded homotopy category of $A\text{-}\mathcal{G}proj$, and by $K^{-,gpb}(A\text{-}\mathcal{G}proj)$ the full subcategory

$$K^{-,gpb}(A\text{-}\mathcal{G}proj) := \{G^\bullet \in K^-(A\text{-}\mathcal{G}proj) \mid \exists \text{ a positive integer } N \text{ such that } H^{-n} \text{Hom}_A(E, G^\bullet) = 0, \forall n > N, \forall E \in A\text{-}\mathcal{G}proj\}.$$

Note from [GZ, Theorem 3.6] that if $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$, then there is a triangle-equivalence $D_{gp}^b(A\text{-mod}) \cong K^{-,gpb}(A\text{-}\mathcal{G}proj)$.

We say that two artin algebras A and B are Gorenstein derived equivalent, if there is a triangle-equivalence $D_{gp}^b(A\text{-mod}) \cong D_{gp}^b(B\text{-mod})$.

Recall from [BH, Be4] that an artin algebra A is of finite Cohen-Macaulay type (simply, CM-type) if there are only finitely many isomorphism classes of finitely-generated indecomposable Gorenstein-projective A -modules. Suppose A is an artin algebra of finite CM-type, and G_1, \dots, G_n are all the pairwise non-isomorphic indecomposable finitely-generated Gorenstein-projective A -modules, and $G = \bigoplus_{1 \leq i \leq n} G_i$. Set $\mathcal{G}p(A) := \text{End}_A(G)^{\text{op}}$, which we call the relative Auslander algebra of A . It is clear that G is an $A\text{-}\mathcal{G}p(A)$ -bimodule and $\mathcal{G}p(A)$ is an artin algebra ([ARS, p.27]). Denote by $\text{mod-}\mathcal{G}p(A)$ the category of left $\mathcal{G}p(A)$ -modules. Recall from [BR, Be3] that an artin algebra A is called virtually Gorenstein if $A\text{-}\mathcal{GP}^\perp = {}^\perp A\text{-}\mathcal{GI}$.

Let A be an artin algebra. Denote by $A\text{-}\mathcal{G}proj(M, N)$ the subgroup of $\text{Hom}_A(M, N)$ of A -maps from M to N which factors through the finitely-generated Gorenstein-projective modules, and $A\text{-mod}/A\text{-}\mathcal{G}proj$ the quotient category of $A\text{-mod}$ modulo $A\text{-}\mathcal{G}proj$, i.e., the objects are same as those of $A\text{-mod}$, and the morphism space from M to N is the quotient group $\text{Hom}_A(M, N)/A\text{-}\mathcal{G}proj(M, N)$. In the following, we call $A\text{-mod}/A\text{-}\mathcal{G}proj$ the Gorenstein stable category of A .

Following [Be1, Definition 3.1], the stabilization of $A\text{-mod}/A\text{-}\mathcal{G}proj$ is a pair $(S, S(A\text{-mod}/A\text{-}\mathcal{G}proj))$, where $S(A\text{-mod}/A\text{-}\mathcal{G}proj)$ is a triangulated category and $S : A\text{-mod}/A\text{-}\mathcal{G}proj \rightarrow S(A\text{-mod}/A\text{-}\mathcal{G}proj)$ is an exact functor, such that for any exact functor $F : A\text{-mod}/A\text{-}\mathcal{G}proj \rightarrow \mathcal{C}$ to a triangulated category \mathcal{C} , there exists a unique triangle functor $F^* : S(A\text{-mod}/A\text{-}\mathcal{G}proj) \rightarrow \mathcal{C}$ such that $F^*S = F$. For the construction of $S(A\text{-mod}/A\text{-}\mathcal{G}proj)$, see ([He], [Be1]).

3. On Gorenstein derived categories

In this section, we characterize the Gorensteinness of an algebra A in three ways: the existence of Auslander-Reiten triangles in the bounded Gorenstein derived category $D_{gp}^b(A\text{-mod})$ of A ; recollements of $D_{gp}^b(A\text{-mod})$, and also Gorenstein derived equivalences.

Let A be an artin algebra and $D_{sg}(A)$ the singularity category of A . Buchweitz's Theorem ([Buc]) has shown that there is an embedding $F : A\text{-}\mathcal{G}proj \rightarrow D_{sg}(A)$ given by $F(G) = G$, where the second G is the corresponding stalk complex at degree 0, and that if A is Gorenstein, then F is a triangle-equivalence. The converse is also true (cf. Theorem 6.9 in [Be1]). In general, to measure how far a ring is from being Gorenstein, Bergh, Jørgensen and Oppermann ([BJO]) defined the Gorenstein defect category $D_{defect}^b(A) := D_{sg}(A)/\text{Im}F$, and also they have shown that A is Gorenstein if and only if $D_{defect}^b(A) = 0$.

To recognize Gorenstein rings via Gorenstein derived categories, we compare a Gorenstein defect category with a Gorenstein derived category, and try to construct a precise relation. We start with the following lemma.

Lemma 3.1. *Let A be an artin algebra such that $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$. Then there is a triangle-equivalence*

$$D_{defect}^b(A) \cong D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj).$$

Proof. By [KZ, Final Remark] we have a triangle-equivalence

$$D_{defect}^b(A) \cong K^{-gp}(A\text{-}\mathcal{G}proj)/K^b(A\text{-}\mathcal{G}proj).$$

By [GZ, Theorem 3.6] we get that

$$D_{gp}^b(A\text{-mod}) \cong K^{-gp}(A\text{-}\mathcal{G}proj).$$

This completes the proof.

Now we test the Gorensteinness of A by the structures of $D_{gp}^b(A\text{-mod})$.

Corollary 3.2. *Let A be an artin algebra such that $A\text{-}\mathcal{G}proj$ is contravariantly finite. Then A is Gorenstein if and only if there is a triangle-equivalence $D_{gp}^b(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}proj)$.*

Proof. By [GZ, Corollary 3.8] we only need to verify the sufficiency. Since $D_{gp}^b(A\text{-mod}) \cong K^b(A\text{-}\mathcal{G}proj)$, it follows that $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj) = 0$. By Lemma 3.1 we get that $D_{defect}^b(A) = 0$. [BJO, Theorem 2.8] implies A is Gorenstein. ■

Corollary 3.3. *Let A be a virtually Gorenstein algebra or of finite CM-type. Then $D_{gp}^b(A\text{-mod})$ has Auslander-Reiten triangles if and only if A is Gorenstein.*

Proof. First we define the functor $\phi : A\text{-}\mathcal{G}proj \rightarrow \text{mod}(A\text{-}\mathcal{G}proj)\text{-proj}$ by $\phi(G) := \text{Hom}_A(-, G)$ for all $G \in A\text{-}\mathcal{G}proj$. Then by Yoneda Lemma we know that ϕ is an equivalence. This follows that $K^b(A\text{-}\mathcal{G}proj) \cong K^b(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj})$ and $K^{-,gp}(A\text{-}\mathcal{G}proj) \cong K^{-,b}(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj})$.

Whenever A is a virtually Gorenstein algebra or of finite CM-type, by [Be4] $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$. This follows from [GZ, Theorem 3.6] that $D_{gp}^b(A\text{-mod}) \cong K^{-,gp}(A\text{-}\mathcal{G}proj)$. By Corollary 3.2 we get that A is Gorenstein if and only if there is a triangle-equivalence $K^{-,b}(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj}) \cong K^b(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj})$. By [Ha3] we get that $K^{-,b}(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj})$ has Auslander-Reiten triangles if and only if the category $\text{mod}(A\text{-}\mathcal{G}proj)$ has finite global dimension if and only if $K^{-,b}(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj}) \cong K^b(\text{mod}(A\text{-}\mathcal{G}proj)\text{-proj})$. Hence $D_{gp}^b(A\text{-mod})$ has Auslander-Reiten triangles if and only if A is Gorenstein by above arguments. ■

Naturally there are two ways to compare Gorenstein derived categories. One way is by Gorenstein derived equivalence. Another way is by recollements of Gorenstein derived categories. We first show the Gorensteinness is invariant under Gorenstein derived equivalences.

Theorem 3.4. *Let A and B be two artin algebras such that $A\text{-}\mathcal{G}proj$ and $B\text{-}\mathcal{G}proj$ are contravariantly finite respectively. If A and B are Gorenstein derived equivalent, then A is Gorenstein if and only if B is Gorenstein.*

Proof. Since A and B are Gorenstein derived equivalent, it follows from [GZ, Proposition 4.2] that there is a triangle-equivalence

$$K^b(A\text{-}\mathcal{G}proj) \cong K^b(B\text{-}\mathcal{G}proj).$$

Thus by Lemma 3.1 we get a triangle-equivalence

$$D_{defect}^b(A) \cong D_{defect}^b(B).$$

This implies that A is Gorenstein if and only if B is Gorenstein. ■

Let A be an algebra of finite CM-type over a commutative artin ring R . Not much is known about its relative Auslander algebra. Next we will show that the torsionless-finiteness is invariant under stable equivalences. We first need the following lemmas.

Lemma 3.5. *Let A be an artin R -algebra of finite CM-type. Then $A\text{-}\mathcal{G}proj$ is a dualizing R -variety, also $\text{mod}(A\text{-}\mathcal{G}proj)$ is a dualizing R -variety.*

Proof. The fact that A is an artin algebra shows that $A\text{-}\mathcal{G}proj$ is a finite R -variety. Since $\mathcal{G}p(A)$ is an artin ring, we know that $\text{mod-}\mathcal{G}p(A)$ is an abelian category. Hence by isomorphisms $\text{mod}(A\text{-}\mathcal{G}proj) \cong \text{mod}(\mathcal{G}p(A)\text{-proj}) \cong \text{mod-}\mathcal{G}p(A)$ we get that $\text{mod}(A\text{-}\mathcal{G}proj)$ is abelian and so $A\text{-}\mathcal{G}proj$ has pseudokernels. Next, suppose E is in $A\text{-}\mathcal{G}proj$. Then the object G in $\text{mod}(A\text{-}\mathcal{G}proj)$ obviously has the property that $\text{Hom}_A(X, E) \cong \text{Hom}_{\mathcal{G}p(A)}(\text{Hom}_A(G, X), \text{Hom}_A(G, E))$ for all X in $\text{mod}(A\text{-}\mathcal{G}proj)$. In order to finish the proof we observe that $(A\text{-}\mathcal{G}proj)^{\text{op}}$ is equivalent to $A^{\text{op}}\text{-}\mathcal{G}proj$ by means of the duality $A\text{-}\mathcal{G}proj \rightarrow A^{\text{op}}\text{-}\mathcal{G}proj$ given by $E \mapsto \text{Hom}_A(E, A)$ for all E in $A\text{-}\mathcal{G}proj$. Since A^{op} is also an artin algebra with center R , it follows that $(A\text{-}\mathcal{G}proj)^{\text{op}} = A^{\text{op}}\text{-}\mathcal{G}proj$ has the properties just derived for $A\text{-}\mathcal{G}proj$. Therefore $A\text{-}\mathcal{G}proj$ satisfies the conditions of Theorem 2.4 in [AR] and so $A\text{-}\mathcal{G}proj$ is a dualizing R -variety. This implies that $\text{mod}(A\text{-}\mathcal{G}proj)$ is also a dualizing R -variety by [AR, Proposition 2.6]. ■

Lemma 3.6. *Let A and B be two artin R -algebras of finite CM-type. If $\mathcal{G}p(A)$ and $\mathcal{G}p(B)$ are stably equivalent, then*

- (1) *there is a bijection between the isomorphism classes of indecomposable non-injective torsionless modules in $\text{mod-}\mathcal{G}p(A)$ and those in $\text{mod-}\mathcal{G}p(B)$.*
- (2) *there is a bijection between the isomorphism classes of indecomposable nonsimple non-projective injective modules in $\text{mod-}\mathcal{G}p(A)$ and those in $\text{mod-}\mathcal{G}p(B)$.*

Proof. Since A and B are of finite CM-type, it follows from Lemma 3.4 that $A\text{-}\mathcal{G}proj$ and $B\text{-}\mathcal{G}proj$ are dualizing R -varieties. Since $\mathcal{G}p(A)$ and $\mathcal{G}p(B)$ are stably equivalent, we have that $A\text{-}\mathcal{G}proj$ and $B\text{-}\mathcal{G}proj$ are two stably equivalent dualizing R -varieties. The result follows from [AR, Corollary 9.11 and 9.14].

Following Ringel [Ri], an artin algebra A is torsionless-finite provided there are only finitely many isomorphism classes of indecomposable torsionless A -modules.

Theorem 3.7. *Let A and B be two artin R -algebras of finite CM-type. If $A\text{-mod}/A\text{-}\mathcal{G}proj$ and $B\text{-mod}/B\text{-}\mathcal{G}proj$ are equivalent as categories, then $\mathcal{G}p(A)$ is torsionless-finite if and only if $\mathcal{G}p(B)$ is torsionless-finite.*

Proof. By [G1] we know that if $A\text{-mod}/A\text{-}\mathcal{G}proj$ and $B\text{-mod}/B\text{-}\mathcal{G}proj$ are equivalent as categories, then $\mathcal{G}p(A)$ and $\mathcal{G}p(B)$ are stably equivalent. By Lemma 3.6 we immediately deduce that $\mathcal{G}p(A)$ is torsionless-finite if and only if $\mathcal{G}p(B)$ is torsionless-finite.

Next, we will compare the invariants along recollements of Gorenstein derived categories. We show the Gorensteinness, the finiteness of Cohen-Macaulay type and finistic dimension are invariant with respect to such recollements. We need the following lemma, which is inspired by [AKLY].

Lemma 3.8. *Let A and B be two artin algebras and $F : K^{-,gpb}(A\text{-}\mathcal{G}proj) \rightarrow K^{-,gpb}(B\text{-}\mathcal{G}proj)$ a triangle functor. The following conditions are equivalent*

- (1) F restricts to $K^b(A\text{-}\mathcal{G}proj)$;
- (2) $F(G) \in K^b(B\text{-}\mathcal{G}proj)$ for any $G \in A\text{-}\mathcal{G}proj$.

Theorem 3.9. *Let A , B and C be virtually Gorenstein algebras. Assume that $D_{gp}^b(A\text{-mod})$ admits the following recollement*

$$D_{gp}^b(B\text{-mod}) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} D_{gp}^b(A\text{-mod}) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} D_{gp}^b(C\text{-mod})$$

Then

- (1) A is of finite CM-type if and only if B and C are so;
- (2) A is Gorenstein if and only if B and C are so.

Proof. Since A is virtually Gorenstein, it follows from [Be4] that $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$. This implies that $D_{gp}^b(A) \cong K^{-,gpb}(A\text{-}\mathcal{G}proj)$ by the proof of [GZ, Theorem 3.6(ii)]. Similar for B and C . By the proof of [GZ, Proposition 4.2 and Lemma 4.3] we get that $i^* : D_{gp}^b(A\text{-mod}) \rightarrow D_{gp}^b(B\text{-mod})$ and $j_! : D_{gp}^b(C\text{-mod}) \rightarrow D_{gp}^b(A\text{-mod})$ can restrict to $K^b(A\text{-}\mathcal{G}proj)$ and $K^b(C\text{-}\mathcal{G}proj)$ respectively. since (i^*, i_*) is an adjoint pair and i^* is a functor from $D_{gp}^b(A\text{-mod})$ to $D_{gp}^b(B\text{-mod})$, we easily get that i_* restricts to $K^b(A\text{-}\mathcal{G}proj)$.

If A is of finite CM-type, then by Lemma 3.8 and [Ni] $i^*(G_A) \in K^b(B\text{-}\mathcal{G}proj)$ is a generator of $D_{gp}^b(B\text{-mod})$. This implies that B is of finite CM-type. For the converse, by above arguments and Lemma 3.8 we get that $i_*(G_B)$ and $j_!(G_C)$ are in $K^b(A\text{-}\mathcal{G}proj)$. Notice from [Ni] that $i_*(G_B)$ and $j_!(G_C)$ generate $D_{gp}^b(A)$. This follows that A is of finite CM-type.

If A is Gorenstein, then by [GZ, Corollary 3.8] we have that $D_{gp}^b(A) \cong K^b(A\text{-}\mathcal{G}proj)$. Since $i^*(G) \in K^b(B\text{-}\mathcal{G}proj)$ for any $G \in A\text{-}\mathcal{G}proj$ by Lemma 3.8, we get that every finitely-generated B -module M admits a proper Gorenstein-projective resolution of finite length. This means $D_{gp}^b(B) \cong K^b(B\text{-}\mathcal{G}proj)$. Hence by Corollary 3.2 we get that B is Gorenstein. For the converse, if B and C are Gorenstein, then we can similarly prove that $D_{gp}^b(A) \cong K^b(A\text{-}\mathcal{G}proj)$. This implies that A is Gorenstein. \blacksquare

Now we will show the finiteness of finistic dimension is invariant with respect to the recollements of bounded Gorenstein derived categories.

Lemma 3.10. *Let A be an artin of finite CM-type. Then $\text{fd}(A) < \infty$ if and only if $\text{fd}(\mathcal{G}p(A)) < \infty$.*

Proof. Let $\text{fd}(A) < \infty$. Let $_{\mathcal{G}p(A)}M$ be a $\mathcal{G}p(A)$ -module of finite projective dimension and let $_{\mathcal{G}p(A)}N$ be an arbitrary $\mathcal{G}p(A)$ -module. Let P_M^\bullet and P_N^\bullet be projective resolutions of M and N . Note that by assumption $P_M^\bullet \in K^b(\mathcal{G}p(A)\text{-proj})$ and $P_N^\bullet \in K^{-,b}(\mathcal{G}p(A)\text{-proj})$.

Then we have isomorphisms for all $n \in \mathbb{N}$

$$\begin{aligned}
\text{Ext}_{\mathcal{G}p(A)}^n(M, N) &= \text{Hom}_{D^b(\mathcal{G}p(A)\text{-mod})}(M, N[n]) \\
&\cong \text{Hom}_{K^{-,b}(\mathcal{G}p(A)\text{-proj})}(P_M^\bullet, P_N^\bullet[n]) \\
&\cong \text{Hom}_{K^{-,gpb}(A\text{-}\mathcal{G}proj)}(G \otimes_{\mathcal{G}p(A)} P_M^\bullet, G \otimes_{\mathcal{G}p(A)} P_N^\bullet[n]) \\
&\cong \text{Hom}_{D_{gp}^b(A\text{-mod})}(G \otimes_{\mathcal{G}p(A)} P_M^\bullet, G \otimes_{\mathcal{G}p(A)} P_N^\bullet[n]).
\end{aligned}$$

Consider the complexes $E_1^\bullet = (E_1^i, d^i) = G \otimes_{\mathcal{G}p(A)} P_M^\bullet$ and $E_2^\bullet = G \otimes_{\mathcal{G}p(A)} P_N^\bullet$. Note that $E_2^i = 0$ for $i > 0$ and that $H^{-s}\text{Hom}_A(G, E_1^\bullet) = 0$ for $s > 1$. This follows that $H^{-s}\text{Hom}_A(G', E_1^\bullet) = 0$ for $s > 1$ and each Gorenstein-projective module G' . Set

$$X^\bullet := \cdots \rightarrow 0 \rightarrow \text{Im}d^{-3} \rightarrow E_1^{-2} \rightarrow E_1^{-1} \rightarrow E_1^0 \rightarrow 0 \rightarrow \cdots.$$

in $D_{gp}^b(A\text{-mod})$ such that $X^\bullet \cong E_1^\bullet$. Since $X^\bullet \cong E_1^\bullet \in K^b(A\text{-}\mathcal{G}proj)$, we infer that $\mathcal{GPdim}_A \text{Im}d^{-3} < \infty$. Therefore $X^\bullet \cong E^\bullet \in K^b(A\text{-}\mathcal{G}proj)$ with width $w(E^\bullet) \leq \text{fGd}(A) + 4$. Thus E^\bullet and $E_2^\bullet[n]$ have disjoint support for $n > \text{fGd}(A) + 4$. The isomorphisms above then shows that $\text{Ext}_{\mathcal{G}p(A)}^n(M, N) = 0$ for $n > \text{fGd}(A) + 4$, so $\text{Ext}_{\mathcal{G}p(A)}^n(M, N) = 0$ for $n > \text{fd}(A) + 4$ by Theorem 2.28 in [Hol]. This implies $\text{fd}(\mathcal{G}p(A)) < \infty$.

Let $\text{fd}(\mathcal{G}p(A)) < \infty$. Let G_M^\bullet and G_N^\bullet be proper Gorenstein-projective resolutions of arbitrary A -modules M and N . Then we have isomorphisms for $n \in \mathbb{Z}$

$$\begin{aligned}
\text{Ext}_{\mathcal{G}}^n(M, N) &= \text{Ext}_{A\text{-}\mathcal{G}p}^n(M, N) \\
&= \text{Hom}_{D_{gp}^b(A\text{-mod})}(M, N[n]) \\
&\cong \text{Hom}_{D_{gp}^b(A\text{-mod})}(G_M^\bullet, G_N^\bullet[n]) \\
&\cong \text{Hom}_{D^b(\mathcal{G}p(A)\text{-mod})}(\text{Hom}_A(G, G_M^\bullet), \text{Hom}_A(G, G_N^\bullet[n])) \\
&\cong \text{Hom}_{D^b(\mathcal{G}p(A)\text{-mod})}(\text{Hom}_A(G, M), \text{Hom}_A(G, N[n])) \\
&\cong \text{Ext}_{\mathcal{G}p(A)}^n(\text{Hom}_A(G, M), \text{Hom}_A(G, N))
\end{aligned}$$

Since $\text{fd}(\mathcal{G}p(A)) < \infty$, we get that $\text{Ext}_{\mathcal{G}}^n(M, N) = 0$ for all $n > \text{fd}(\mathcal{G}p(A))$. This follows that $\text{fd}(A) = \text{fGd}(A) < \infty$ again by Theorem 2.28 in [Hol]. \blacksquare

The following corollary is proved by Asadollahi, Hafezi and Vahed in [AHV]. Now we give an alternative proof.

Corollary 3.11. *Let A and B be artin algebras of finite CM-type. If A and B are Gorenstein derived equivalent, then $\text{fd}(A) < \infty$ if and only if $\text{fd}(B) < \infty$.*

Proof. Since $D_{gp}^b(A\text{-mod}) \cong D_{gp}^b(B\text{-mod})$ by assumption, we get that $\mathcal{G}p(A)$ and $\mathcal{G}p(B)$ are derived equivalent. This follows that $\text{fd}(\mathcal{G}p(A)) < \infty$ if and only if $\text{fd}(\mathcal{G}p(B)) < \infty$ by [PX, Theorem 1.1].

By Lemma 3.10 we get that $\text{fd}(A) < \infty$ if and only if $\text{fd}(\mathcal{G}p(A)) < \infty$, and $\text{fd}(B) < \infty$ if and only if $\text{fd}(\mathcal{G}p(B)) < \infty$. Thus $\text{fd}(A) < \infty$ if and only if $\text{fd}(B) < \infty$. \blacksquare

Theorem 3.12. *Let A be an artin algebra of finite CM-type. Assume that $D_{gp}^b(A\text{-mod})$ has a recollement relative to $D_{gp}^b(B\text{-mod})$ and $D_{gp}^b(C\text{-mod})$ for artin algebras B, C of finite CM-type. Then $\text{fd}A < \infty$ if and only if $\text{fd}B < \infty$ and $\text{fd}C < \infty$.*

Proof. Since A, B and C are artin algebras of finite CM-type, there are triangle-equivalences $D_{gp}^b(A\text{-mod}) \cong D^b(\mathcal{G}p(A)\text{-mod})$, $D_{gp}^b(B\text{-mod}) \cong D^b(\mathcal{G}p(B)\text{-mod})$ and also $D_{gp}^b(C\text{-mod}) \cong D^b(\mathcal{G}p(C)\text{-mod})$. It follows that $D^b(\mathcal{G}p(A)\text{-mod})$ has a recollement relative to $D^b(\mathcal{G}p(B)\text{-mod})$ and $D^b(\mathcal{G}p(C)\text{-mod})$. Hence by Theorem 3.3 in [Ha4] $\text{fd}(\mathcal{G}p(A)) < \infty$ if and only if $\text{fd}(\mathcal{G}p(B)) < \infty$ and $\text{fd}(\mathcal{G}p(C)) < \infty$.

By Lemma 3.10 we get that $\text{fd}A < \infty$ if and only if $\text{fd}(\mathcal{G}p(A)) < \infty$, $\text{fd}B < \infty$ if and only if $\text{fd}(\mathcal{G}p(B)) < \infty$, and $\text{fd}C < \infty$ if and only if $\text{fd}(\mathcal{G}p(C)) < \infty$, respectively. This completes the proof. \blacksquare

4. On Gorenstein defect categories

The previous section used Gorenstein defect categories as a crucial tool. However, not much is known about Gorenstein defect categories. In this section, we will show the Karoubianness of Gorenstein defect categories, and establish relations between Gorenstein defect categories and Gorenstein stable categories.

We first determine the dimension of Gorenstein defect categories for a simple class of algebras.

Example 4.1. *Let A be a representation-finite artin algebra. Then $\dim D_{\text{defect}}^b(A) \leq 1$.*

Proof. Since A is representation-finite, it follows from [O] that $\dim D^b(A) \leq 1$. By [Ro, Lemma 3.4] we get that $\dim D_{\text{defect}}^b(A) \leq \dim D_{\text{sg}}(A) \leq \dim D^b(A)$. Hence $\dim D_{\text{defect}}^b(A) \leq 1$. \blacksquare

Now we study the Karoubianness of the Gorenstein defect category of an algebra. We need some preparation.

Suppose A is of finite CM-type. For any M in $A\text{-mod}/A\text{-}\mathcal{G}proj$, we can take a right $A\text{-}\mathcal{G}proj$ -approximation of M , and denote its kernel by M^1 . Then we have a functor $\overline{\Omega} : A\text{-mod}/A\text{-}\mathcal{G}proj \rightarrow A\text{-mod}/A\text{-}\mathcal{G}proj$, view $\overline{\Omega}(M) := M^1$. Denote by $\overline{\Omega}^n$ the n th composition functor of $\overline{\Omega}$ for any positive integer $n \geq 2$. Then we have

Lemma 4.2. *Let X^\bullet be a complex in $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$ and $n_0 > 0$. Then for any n large enough, there exists a module M in $\overline{\Omega}^{n_0}(A\text{-mod})$ such that $X^\bullet \simeq Q(M)[n]$, where $Q : D_{gp}^b(A\text{-mod}) \rightarrow D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$ is the quotient functor.*

Proof. Take an $A\text{-}\mathcal{G}proj$ -quasi-isomorphism $G^\bullet \rightarrow X^\bullet$ with $G^\bullet \in K^-(A\text{-}\mathcal{G}proj)$. Take $n \geq n_0$ such that $H^i \text{Hom}_A(E, X^\bullet) = 0$ for all $i \leq n_0 - n$ and $E \in A\text{-}\mathcal{G}proj$. Consider the truncation $\tau^{\geq -n} G^\bullet = \cdots \rightarrow 0 \rightarrow M \rightarrow G^{1-n} \rightarrow G^{2-n} \rightarrow \cdots$ of G^\bullet , which is $A\text{-}\mathcal{G}proj$ -quasi-isomorphic to G^\bullet . Then the cone of the obvious chain map $\tau^{\geq -n} G^\bullet \rightarrow M[n]$ is in $K^b(A\text{-}\mathcal{G}proj)$, which becomes an isomorphism in $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$. This shows that $X^\bullet \simeq Q(M)[n]$. We observe that M lies in $\overline{\Omega}^{n_0}(A\text{-mod})$. \blacksquare

Lemma 4.3. *Let $0 \rightarrow M \rightarrow G^{1-n} \rightarrow \cdots \rightarrow G^0 \rightarrow N \rightarrow 0$ be an $A\text{-}\mathcal{G}proj$ -acyclic complex with each G^i Gorenstein-projective. Then we have an isomorphism $Q(N) \cong Q(M)[n]$ in $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$. In particular, for an A -module M , we have a natural isomorphism $Q(\overline{\Omega}^n(M)) \cong Q(M)[-n]$.*

Proof. The stalk complex N is $A\text{-}\mathcal{G}proj$ -quasi-isomorphic to $\cdots \rightarrow 0 \rightarrow M \rightarrow G^{1-n} \rightarrow \cdots \rightarrow G^0 \rightarrow 0$. This gives rise to a morphism $N \rightarrow M[n]$ in $D_{gp}^b(A\text{-mod})$, whose cone is in $K^b(A\text{-}\mathcal{G}proj)$. Then this morphism becomes an isomorphism in $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$. \blacksquare

We consider the composite functor $Q' : A\text{-mod} \hookrightarrow D_{gp}^b(A\text{-mod}) \xrightarrow{Q} D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$. It vanishes on $A\text{-}\mathcal{G}proj$, so it induces uniquely a functor $A\text{-mod}/A\text{-}\mathcal{G}proj \rightarrow D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$, which still denote by Q' . Then for any modules M, N in

$A\text{-mod}/A\text{-}\mathcal{G}proj$, the functor Q' induces a natural map by Lemma 4.3

$$\Phi^0 : \text{Hom}_A(M, N)/A\text{-}\mathcal{G}proj(M, N) \rightarrow \text{Hom}_{D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)}(Q(M), Q(N)).$$

This induces the map

$$\Phi^n : \text{Hom}_A(\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}proj(M, N) \rightarrow \text{Hom}_{D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)}(Q(M), Q(N))$$

given by $\Phi^n(f) = (\theta_N)^{-1} \circ (\Phi^0(f)[n]) \circ \theta_M$.

Consider the chain of maps $\text{Hom}_A(\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}proj(M, N) \rightarrow \text{Hom}_A(\overline{\Omega}^{n+1}(M), \overline{\Omega}^{n+1}(N))/A\text{-}\mathcal{G}proj(M, N)$ induced by $\overline{\Omega}$. Then we have an induced map

$$\Phi : \varinjlim \text{Hom}_A((\overline{\Omega}^n(M), \overline{\Omega}^n(N))/A\text{-}\mathcal{G}proj(M, N) \rightarrow \text{Hom}(Q(M), Q(N)).$$

■

Lemma 4.4. *Let A be an artin algebra of finite CM-type, and M, N be in $A\text{-mod}/A\text{-}\mathcal{G}proj$. Then the map Φ is an isomorphism.*

Proof. We refer to [Be1, Theorem 3.8] for a detailed proof. ■

Recall that an additive category \mathcal{A} is Karoubian (i.e. idempotent split) provided that each idempotent $e : X \rightarrow X$ splits, that is, it admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ with $u \circ v = \text{Id}_Y$. In particular, for an artin algebra A , the quotient category $A\text{-mod}/A\text{-}\mathcal{G}proj$ is Karoubian.

Theorem 4.5. *The Gorenstein defect category $D_{defect}^b(A)$ of an artin algebra A of finite CM-type is Karoubian.*

Proof. We claim that $D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$ is Karoubian. By Lemma 4.3 it suffices to show that for each module M in $A\text{-mod}/A\text{-}\mathcal{G}proj$, an idempotent $e : Q(M) \rightarrow Q(M)$ splits. Lemma 4.4 implies that for a large n , there is an idempotent $e^n : \overline{\Omega}^n(M) \rightarrow \overline{\Omega}^n(M)$ in $A\text{-mod}/A\text{-}\mathcal{G}proj$ which is mapped by Φ to e . Note that the idempotent e^n splits. Then the idempotent e splits. By Lemma 3.1 we have a triangle-equivalence $D_{defect}^b(A) \cong D_{gp}^b(A\text{-mod})/K^b(A\text{-}\mathcal{G}proj)$. Hence $D_{defect}^b(A)$ is Karoubian. ■

Next we will show the Gorenstein defect category of an algebra A is triangular equivalent to the stabilization of the Gorenstein stable category of A .

Lemma 4.6. *Let A be an artin algebra such that $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$. Then there is a triangle-equivalence*

$$S(A\text{-mod}/A\text{-}\mathcal{G}proj) \cong D_{defect}^b(A).$$

Proof. Since $A\text{-}\mathcal{G}proj$ is contravariantly finite in $A\text{-mod}$, it follows from [Be1, Theorem 3.8] that there is a triangle-equivalence

$$S(A\text{-mod}/A\text{-}\mathcal{G}proj) \cong K^{-,gp^b}(A\text{-}\mathcal{G}proj)/K^b(A\text{-}\mathcal{G}proj).$$

By [KZ, Finial Remark] we get a triangle-equivalence

$$S(A\text{-mod}/A\text{-}\mathcal{G}proj) \cong D_{defect}^b(A).$$

■

As an application we show the equivalences of Gorenstein stable categories can induce the equivalences of Gorenstein defect categories for two algebras of finite CM-type. For convenience we introduce two definitions.

Definition 4.7. *Two artin algebras A and B are said to be Gorenstein stably equivalent if their Gorenstein stable categories $A\text{-mod}/A\text{-Gproj}$ and $B\text{-mod}/B\text{-Gproj}$ are equivalent as left triangulated categories.*

Definition 4.8. *Two artin algebras A and B are said to be Gorenstein defect equivalent if there is a triangle-equivalence $D_{\text{defect}}^b(A) \cong D_{\text{defect}}^b(B)$.*

Corollary 4.9. *Let A and B be two artin algebras such that $A\text{-Gproj}$ and $B\text{-Gproj}$ are contravariantly finite respectively. If A and B are Gorenstein stable equivalent, then A and B are Gorenstein defect equivalent. Thus the Gorenstein defect category of an algebra of finite CM-type is uniquely determined by its Gorenstein stable category.*

Proof. Since A and B are Gorenstein stably equivalent, then there is a triangle-equivalence by [Be1, Corollary 3.3]

$$S(A\text{-mod}/A\text{-Gproj}) \cong S(B\text{-mod}/B\text{-Gproj}).$$

By Lemma 4.6 we have triangle-equivalences

$$S(A\text{-mod}/A\text{-Gproj}) \cong D_{\text{defect}}^b(A),$$

and

$$S(B\text{-mod}/B\text{-Gproj}) \cong D_{\text{defect}}^b(B).$$

Hence we get a triangle-equivalence

$$D_{\text{defect}}^b(A) \cong D_{\text{defect}}^b(B).$$

■

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